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Abstract

This report discusses different approaches to stable H^∞ controller design applied to the problem of augmenting the longitudinal dynamics of an aircraft. Stability of the H^∞ controller is investigated by analyzing the effects of changes in the performance index weights, and modifications in the measured outputs. The existence of a stable suboptimal controller is also investigated. It is shown that this is equivalent to finding a stable controller, whose infinity norm is less than a specified bound, for an unstable plant which is determined from parametrization of all H^∞ controllers. Examples are given for a gust alleviation and a command tracking problems.

[1]This work was performed while the first author was a NASA/OAI Summer Faculty Fellow at NASA Lewis Research Center.

1 Introduction

The problem of finding a stable feedback controller is known as strong stabilization, and it has been widely studied in the literature, see e.g. [1]. One of the motivations behind this problem is that, when the plant is stable, unstable controllers are not tolerant to faults in measurements. An example of this is when a feedback path is broken, such a controller may lead to an unbounded response for a bounded reference input. Also, from a real-time implementation point of view, it may be undesirable to have an unstable feedback controller, [2], [3].

A necessary and sufficient condition for the existence of a strongly stabilizing controller is the parity interlacing property (p.i.p.) [4]. A plant satisfies p.i.p. if the number of poles between any pairs of distinct blocking zeros on the positive real axis (including $+\infty$) is even. There are procedures for constructing stable controllers which stabilize a given plant, [5, 1, 4]. However, there is no simple parametrization for the set of all strongly stabilizing controllers. Usually, closed loop stability is the first design requirement. But one is also interested in achieving some kind of robustness and performance level in the controller design. This requirement can be achieved (with a certain degree of conservatism) by using $\mathcal{H}^\infty$ control techniques. Therefore, finding strongly stabilizing $\mathcal{H}^\infty$ controllers is an important research problem, which is the subject of this report. It should also be mentioned that some promising results appear in [6] (see also references therein) on the $\mathcal{H}^2$ version of this problem. The effects of weight selection on the stability of the optimal $\mathcal{H}^\infty$ controller for SISO plants have also been studied in [7].

The plant considered in this study is a linear model of the longitudinal dynamics of an experimental F-15 aircraft with pitch vectoring nozzles. For this plant, two $\mathcal{H}^\infty$ control design problems are defined. The first one deals with gust alleviation, and the second one is a command tracking problem. The results of [8], [9] and [10] are used in order to obtain a parametrization of all suboptimal $\mathcal{H}^\infty$ controllers. The underlying operator theoretical results for this parametrization can be found in [11, 12], see also [13] for more details. Most commercially available softwares (e.g. robust control modules of $MATLAB$ and $MATRIX_X$) generate the so-called "central controller" of [9], (see Section 2.2 for the precise definition of the central controller) . Here, the effect of structural changes in the plant (e.g. adding one more output for feedback) on the stability of the central controller is studied first. Then, the effects of scaling the performance index weighting functions, and the effect of increasing (or decreasing) the $\mathcal{H}^\infty$ suboptimal peformance level γ, on the stability of the central controller are studied. Finally, the parametrization of all suboptimal $\mathcal{H}^\infty$ controllers is studied, and several different methods of finding a stable controller in this parametrization are discussed. This is a significant problem, because for a given admissible γ the central controller may be unstable, but there may be a stable controller in the set of all controllers which achieves the same performance level.

The rest of this paper is organized as follows. In the next section a background on the standard $\mathcal{H}^\infty$ control problem is given, along with the formulae for the central controller and the parametrization of all suboptimal controllers. The plant model considered in this paper is also described in the next section. Section 3 is devoted to the studies on the stability of the central controller. In Section 4, stable suboptimal $\mathcal{H}^\infty$ controllers are investigated for the gust alleviation and tracking examples. For both of these examples it is shown that there exists a stable suboptimal controller, while the central controller is unstable. Finally, concluding remarks are made in Section 5.

2 Background on $\mathcal{H}^\infty$ control

2.1 Standard problem set-up

The so-called "standard $\mathcal{H}^\infty$ control problem" deals with the system shown in Figure 1. The system equations are assumed to be given by the following

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \quad (1)$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \quad (2)$$

$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \quad (3)$$

where x represents combined states of the system, and components of w are the exogenous signals (reference inputs, disturbances, measurement noises), components of u are the control inputs, components of y are the measured signals, and components of z are internal signals to be controlled. The optimal $\mathcal{H}^\infty$ problem is to find a feedback controller K (whose input is y and output u) so that the closed loop system is stable, and the worst energy amplification from w to z is minimized. This problem is equivalent to finding a stabilizing controller which minimizes $\|T_{zw}\|_\infty$, where $T_{zw}(s)$ is the closed loop transfer function from w to z. The suboptimal

$\mathcal{H}^\infty$ control problem is to find a stabilizing controller so that $\|T_{zw}\|_\infty < \gamma$, for a specified performance level γ.

Usually, an $\mathcal{H}^\infty$ control problem is first transformed to the above standard form. Then, the controller is obtained from the matrices A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}, D_{22}, using algebraic Riccati equation solvers. The controller formulae are given in Section 2.2.

2.1.1 Aircraft Model

In order to demonstrate how one sets up an $\mathcal{H}^\infty$ control problem, two examples will be considered. Both of these examples involve a nominal plant, which is a linear model for the longitudinal dynamics of an aircraft. The plant is decribed by

$$\begin{aligned} \dot{x}_p(t) &= A_p x_p(t) + B_p u_p(t) \\ y_p(t) &= C_p x_p(t) \end{aligned}$$

where x_p are the states, u_p denotes the command input and y_p denotes the plant output, and

$$A_p = \begin{bmatrix} -0.2201 & -33.6053 & 0.0000 & -25.9641 \\ -0.0014 & -0.2070 & 1.0000 & 0.0630 \\ -0.0015 & 0.4846 & -0.2448 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix} \qquad B_p = \begin{bmatrix} -0.2403 \\ -0.0016 \\ -0.0634 \\ 0.0000 \end{bmatrix}$$

$$C_p = \begin{bmatrix} C_\alpha \\ C_{qp} \end{bmatrix} \quad \text{where} \quad C_\alpha = 57.2958[\,0 \;\; 1 \;\; 0 \;\; 0\,] \;, \quad C_{qp} = 57.2958[\,0 \;\; 0 \;\; 1 \;\; 0\,] \,.$$

The first component of $y_p(t)$ is the angle of attack $\alpha(t)$ (in degrees) and its second component is the pitch angular rate $q_p(t)$ (in degrees/sec). The command input u_p is the nozzle pitch vector angle (in degrees). The components of the states x_p of the airframe are forward velocity (in ft/sec), angle of attack (in radians), pitch rate (in rads/sec), and pitch attitude θ (in rads), respectively. In this system the command signal u_p is the output of an actuator whose dynamical behavior is decribed by

$$\begin{aligned} \dot{x}_a(t) &= -25x_a(t) + 25u(t) \\ u_p(t) &= x_a(t) , \end{aligned}$$

where $u(t)$ denotes the command signal to be generated by the feedback controller.

The poles of the transfer function T_{u_p,y_p}, from u_p to y_p, are

$$[+0.634, \ -0.8647, \ -0.2206 \pm j0.1274] ,$$

and the zeros of the first component $T_{u_p,\alpha}$ and second component T_{u_p,q_p}, of T_{u_p,y_p} are

$$[-39.6, \ -0.1382 \pm j0.1726] \quad \text{and} \quad [-0.436556, \ 0.000, \ +0.002912]$$

respectively. Note that this plant satisfies the parity interlacing property, but the transfer function from u_p to q_p does not. In other words, there exists a strongly stabilizing controller for this plant. But, if the first output $\alpha(t)$ is not used as a control feedback, then for the resulting plant all stabilizing controllers are unstable.

2.1.2 Tracking Problem Definition

Now the $\mathcal{H}^\infty$ control problem, associated with tracking of α-command signal, can be defined in terms of the system shown in Figure 2, where $W_S(s)$ is the sensitivity weight, and $W_T(s)$ is the weight for the complementary sensitivity. Usually W_S is chosen as a low pass filter, and W_T is chosen as a high pass filter. This means that low frequency reference inputs are considered for tracking, and high frequency unmodelled dynamics are taken into account for robustness, see [5]. Accordingly following weights are chosen in this problem

$$W_S(s) = C_s(s - A_s)^{-1}B_s \quad \text{where} \quad A_s = -1/335, \quad B_s = 1000/335, \quad C_s = 1,$$

$$W_T(s) = C_t(s - A_t)^{-1}B_t + D_t \quad \text{where} \quad A_t = -1/.00278, \quad B_t = -99.999/.00278, \quad C_t = 1,$$

and $D_t = 100$, with $R_1 = 1/3$, $R_2 = 1/3$, $R_3 = 0.2$, $R_4 = 0.02$ $K_1 = 3$, $K_2 = 0.001$. The scalars $R_1, \ldots, R_4$ assign relative weights on the internal signals of interest and K_1, K_2 scale the measurement noises. For example if the size of u_p has to be made small then R_3 (weight on u_p) is chosen relatively large. In Figure 2, w_1 represents the reference input (we want α to follow w_1) and $K_2 w_2$ represents a small amount of noise which is present in the measurement of q_p. If $\hat{y} := \begin{bmatrix} y_1/K_1 \\ y_2/K_2 \end{bmatrix}$ and $\hat{u} := 25R_4 u$, then the state equations, corresponding to the generalized system from $[w \;\; \hat{u}]$ to $[z \;\; \hat{y}]$, are in the form (1–3) with

$$A = \begin{bmatrix} -25 & 0 & 0 & 0 \\ B_p & A_p & 0 & 0 \\ 0 & -B_s C_\alpha & A_s & 0 \\ 0 & B_t C_\alpha & 0 & A_t \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ B_s K_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1/R_4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & R_1C_s & 0 \\ 0 & R_2D_tC_\alpha & 0 & R_2C_t \\ R_3 & 0 & 0 & 0 \\ -25R_4 & 0 & 0 & 0 \end{bmatrix} \quad D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & -C_\alpha/K_1 & 0 & 0 \\ 0 & C_{qp}/K_2 & 0 & 0 \end{bmatrix} \quad D_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that the controller generates $u = K(s)y$. But first the controller which generates $\hat{u} = \hat{K}(s)\hat{y}$, can be found and then from $\hat{K}(s)$ one can determine $K(s) = (25R_4)^{-1}\hat{K}(s)\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}^{-1}$. Numerically it is simpler to find $\hat{K}$, because in this set-up D_{12} and D_{21} are normalized to have entries 1 and 0 only. See the formulae in [9] for a comparison.

2.1.3 Gust Alleviation Problem

Consider the plant described above with gust affecting the system dynamics as follows

$$\begin{aligned} \dot{x}_{pg}(t) &= A_{pg}x_{pg}(t) + B_g\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + B_{pg}u_p(t) \\ y_p(t) &= C_{pg}x_{pg}(t) \end{aligned}$$

where $C_{pg} = \begin{bmatrix} 0_{1\times3} & C_\alpha \\ 0_{1\times3} & C_{pq} \end{bmatrix} =: \begin{bmatrix} C_{\alpha g} \\ C_{qpg} \end{bmatrix}$, in other words $y_p = \begin{bmatrix} \alpha \\ q_p \end{bmatrix}$, and

$$A_{pg} = \begin{bmatrix} A_{11g} & 0 \\ A_{21g} & A_p \end{bmatrix} \qquad B_{pg} = \begin{bmatrix} 0 \\ B_p \end{bmatrix}$$

$$B_g = \sigma\sqrt{\frac{2V}{L}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 1 \\ 1 & 0 \\ 0 & \sqrt{3}/V \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_{11g} = -\frac{V}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad A_{21g} = -\frac{1}{L}\begin{bmatrix} V & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $V = 300\text{ft/sec}$, $L = 1670\text{ft}$, and $0 < \sigma < 10$, (a moderate gust, $\sigma = 5$, will be considered). The gust model in the above formulae is the Dryden model, which is described in [14]. The command signal u_p is generated by an actuator whose input is u, which is to be generated by a feedback controller: $u_p = W_A u$, where $W_A(s) = C_a(s - A_a)^{-1}B_a$, with $A_a = -25$, $B_a = 5$ and $C_a = 5$. The $\mathcal{H}^\infty$ problem associated with this system is to minimize the effect of $w_1, \ldots, w_4$ on $z_1, \ldots z_4$ as shown in Figure 3, where w_3/K_1 and w_4/K_2 represent measurement noises, z_1 and z_2 are scaled values of α and q_p respectively, and z_3 is the scaled control, and z_4 is a blend of the control and the control rate.

For this problem define $\hat{y} := \begin{bmatrix} K_1 y_1 \\ K_2 y_2 \end{bmatrix}$, then the system equations which represent the transfer function from $[w, u]$ to $[z, \hat{y}]$ are in the form (1–3), with

$$A = \begin{bmatrix} A_a & 0 \\ B_{pg}C_a & A_{pg} \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 & 0 \\ B_g & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} B_a \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & R_1 C_{\alpha g} \\ 0 & R_2 C_{qpg} \\ R_3 C_a & 0 \\ 0 & 0 \end{bmatrix} \qquad D_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & K_1 C_{\alpha g} \\ 0 & K_2 C_{qpg} \end{bmatrix} \qquad D_{21} = \begin{bmatrix} 0_{2\times 2} & \mathrm{I}_{2\times 2} \end{bmatrix}$$

and $D_{11} = 0_{4\times 8}$, $D_{22} = 0_{2\times 1}$. Similar to the previous case, once a controller $\hat{K}$, which generates u from $\hat{y}$, is determined, the equivalent controller K, which generates the same control u from y, can be found by setting $K(s) = \hat{K}(s) \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$.

2.2 Controller Formulae

The formulae of [8, 9, 10] for $\mathcal{H}^\infty$ controllers, which satisfy a certain specified performance level γ, is given below. The problem formulation, in both tracking and gust alleviation, satisfies the structural assumptions of [9]. The controller expression is simpler if the formulation satisfies the structural assumptions of [8]:

Assumptions

A1. (A, B_2, C_2) is stabilizable and detectable,

A2. $D_{11} = 0$, $D_{22} = 0$,

A3. $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$,

A4. $\qquad D_{12}^T\,[C_1 \quad D_{12}] = [0 \quad I].$

The gust problem defined above satisfies these assumptions. For the tracking problem the orthogonality assumptions **A3** and **A4** are not satisfied, therefore for this example a slightly more complicated formula, [9], will be used.

Now consider the problem formulation given in (1-3), and assume that they satisfy **A1**, ..., **A4**, e.g. the gust alleviation problem is in this form. Then, set up the following Algebraic Riccati Equations (AREs)

$$A^TX + XA + X(\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 = 0 \tag{4}$$

$$AY + YA^T + Y(\frac{1}{\gamma^2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T = 0\,. \tag{5}$$

It has been shown that, [8, 9], there exists a stabilizing controller which makes $||T_{wz}||_\infty < \gamma$ if and only if there exist unique solutions X and Y, for (4) and (5), such that

(i) X and Y are positive semi-definite,

(ii) eigenvalues of $A_X := A + (\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T)X$ and $A_Y := A + Y(\frac{1}{\gamma^2}C_1^TC_1 - C_2^TC_2)$ are in the open left half plane, and

(iii) the largest eigenvalue of XY is strictly less than γ^2 .

Moreover, if (i)–(iii) hold then all controllers, $\hat{K}$, which stabilize the closed loop system and achieve $||T_{zw}||_\infty < \gamma$ are parametrized as follows

$$\dot{x}_c(t) \quad = \quad A_cx_c(t) + B_{1c}\hat{y}(t) + B_{2c}q(t) \tag{6}$$

$$\hat{u}(t) = C_{1c}x_c(t) + q(t) \tag{7}$$

$$r(t) = C_{2c}x_c(t) + \hat{y}(t) \tag{8}$$

where

$$A_c = A_X - ZYC_2^TC_2, \tag{9}$$

$$B_{1c} = ZYC_2^T, \tag{10}$$

$$B_{2c} = ZB_2, \tag{11}$$

$$C_{1c} = -B_2^TX, \tag{12}$$

$$C_{2c} = -C_2 \tag{13}$$

and $q(t)$ is generated by a transfer matrix $Q(s)$, whose input is $r(t)$, with $Q \in \mathcal{H}^\infty$ (that is Q must be stable) and $\|Q\|_\infty < \gamma$. Obviously, there are infinitely many choices for $Q(s)$ and hence there are infinitely many suboptimal controllers. Implementation of this controller is shown in Figure 4. In particular, one can choose $Q(s) = 0$, this gives the "central" controller. The issue of selecting Q will be discussed in Section 4. In [9] and [10] an alternative expression for suboptimal controllers is given. This expression is in the same form as (6–8), but its state space realization is slightly different

$$A_c = A_Y - B_2B_2^TXZ, \tag{14}$$

$$B_{1c} = YC_2^T, \tag{15}$$

$$B_{2c} = B_2, \tag{16}$$

$$C_{1c} = -B_2^TXZ, \tag{17}$$

$$C_{2c} = -C_2 Z. \tag{18}$$

The above realization (14–18) can be obtained via a state transformation on (9–13).

It should be mentioned that for the tracking problem, the parametrization of all suboptimal controllers is exactly the same as (6–8), but the AREs associated with this problem, and hence computations of A_c, B_{1c}, B_{2c}, C_{1c} and C_{2c}, are slightly more complicated, see [9] for all the details.

Note that the central controller is stable if and only if A_c is a stable matrix (i.e. all the eigenvalues of A_c have strictly negative real parts). Whereas, stability of a controller which is obtained from a non-zero $Q(s)$ depends on A_c, B_{2c}, C_{2c} and $Q(s)$, to be discussed further in Section 4.

3 Stability of the central controller

In this section the effects of structural changes and weight scalings, on the stability of the central controller, will be discussed.

3.1 Changes in the output variables

Consider the tracking problem defined in Section 2.1.2. The central controller corresponding to $\gamma = 2$ is unstable (for this system the optimal value of γ is between 1.5 and 2). It will be shown that by changing one of the output variables it is possible to transform the problem to a form where it is easy to analyze stability of the central controller. For this purpose replace the second output $y_2 = q_p + K_2 w_2$ with $y_2 = u_p - \alpha$ where $u_p = C_a x_a + K_2 w_2$. Here $K_2 w_2$ represents a small amount of actuator noise, whereas in the previous case it was a measurement noise. Also for simplicity redefine z_4 from $R_4 u$ to $z_4 = u$. Then, the generalized plant is as shown in Figure 5.

As before, if $\hat{y}_i := y_i / K_i$, for $i = 1, 2$, then the system equations are as in (1–3) where

$$A = \begin{bmatrix} A_a & 0 & 0 & 0 \\ B_p C_a & A_p & 0 & 0 \\ 0 & -B_s C_\alpha & A_s & 0 \\ 0 & B_t C_\alpha & 0 & A_t \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & B_p K_2 \\ B_s K_1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} B_a \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & C_s & 0 \\ 0 & D_t C_\alpha & 0 & C_t \\ C_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0 & -C_\alpha / K_1 & 0 & 0 \\ C_a / K_2 & -C_\alpha / K_2 & 0 & 0 \end{bmatrix}$$

and $D_{12} = [0 \;\; 0 \;\; 0 \;\; 1]^T$, $D_{22} = I$, $D_{11} = 0$, $D_{22} = 0$. It is easy to check that if below assumptions (dff1–dff3) are satisfied then the problem formulation is in the form of the disturbance feedforward (dff) problem defined in [8],

dff1. $C_a B_a$ is invertible, $D_t \neq 0$, $K_2 \neq 0$,

dff2. (A_p, B_p, C_α) is stabilizable and detectable,

dff3. A_a, A_s, A_t and $(A_p + B_p C_\alpha)$ are stable.

In this case the A-matrix of the central controller is

$$A_c := A - B_1 C_2 - B_2 B_2^T X.$$

For the plant shown in Figure 5, the above dff assumptions are satisfied. Indeed one can check that for A_p, B_p and C_α given above the eigenvalues of $(A_p + B_p C_\alpha)$ are at $-0.1221 \pm j0.1757$ and $-0.2597 \pm j1.7419$. It is also easy to verify that in this case A_c is in the form

$$A_c = \begin{bmatrix} A_a - B_a B_a^T X_{11} & \aleph & \aleph & \aleph \\ 0 & A_p + B_p C_\alpha & 0 & 0 \\ 0 & 0 & A_s & 0 \\ 0 & B_t C_\alpha & 0 & A_t \end{bmatrix}$$

where X_{11} is the $1,1$ entry of X, and $\aleph$ represents an entry whose value is not important. By partitioning this matrix one can easily see that the eigenvalues of A_c are the eigenvalues of its diagonal blocks. Since $A_p + B_p C_\alpha$, A_s, A_t, A_a are stable and $B_a B_a^T X_{11}$ is a positive scalar, one concludes that the central controller in this case is stable.

The purpose of this example was to show that by modifying one of the outputs it was possible to obtain a central controller whose A-matrix A_c has a certain special structure for which the stability analysis is easy. Of course for more complicated examples it may be difficult to see which way the output should be modified to get a stable controller. Moreover, in some cases it may be impossible to modify the output physically.

3.2 Scaling the weights and adjusting γ

Now the effects of weight scalings on the stability of the central controller will be studied for the gust alleviation example shown in Figure 3. Note that in this example one can think of K_1, K_2, R_1, R_2, R_3, R_4 and σ as scaling parameters. As illustrated by Figures 6 and 9 through 11, changes in K_1, R_1, R_2 and R_3 do not significantly affect the stability of the central controller. These figures are generated with the nominal values $K_1 = R_1 = R_2 = R_3 = 1$, $K_2 = 0.01$ and $\sigma = 5$. It is seen from Figures 7 and 8 that with these nominal values, if K_2 is larger than 0.1, the central controller is unstable, for both $\gamma = 25$ and $\gamma = 30$. In the above analyses, large enough γ values are chosen in order to guarantee the existence of an H^∞ controller for the parameter variations considered in each plot shown in Figures 6 through 11.

Now fix $K_1 = K_2 = R_1 = R_2 = R_3 = 1$ and $\sigma = 5$. For these values of the scaling parameters

$$18 < \gamma_{opt} < 19 ,$$

and the eigenvalues of the central controller for two different values of γ are given below.

$\gamma = 19$	$\gamma = 25$
$-0.16 \pm j0.06$	$-0.17 \pm j0.053$
$-0.18 \pm j0.0057$	$-0.18 \pm j0.00567$
$\mathbf{+\ 0.6}$	$\mathbf{+\ 0.3}$
-10.5	-5.5
-16.8	-17.1
-34.3	-34.9

This example shows that by increasing the value of γ it is possible to move the unstable pole slightly towards the left half plane. But the central controller remained unstable even for γ as large as 80. In fact, it is known that [8], as $\gamma \to \infty$ the central $\mathcal{H}^\infty$ controller converges to the optimal $\mathcal{H}^2$ controller, which does not have to be stable.

In the tracking problem, the general rule of thumb is that if one is willing to give up the performance to gain more robustness, then one should scale the sensitivity weight W_S by $1/k$ and complementary sensitivity weight by $k > 1$. Then, for k sufficiently large the central controller is expected to be stable, see [7] for a delay system example. To illustrate this point, now consider the generalized plant of Figure 12, which is a slightly different version of the tracking problem defined in Section 2.1.2, (there are two more outputs for feedback, α and θ, with slight additive noise). For this problem when $k = 1$ the optimal value of γ is between 1.5 and 1.75, and for $\gamma = 1.75$ the central controller is unstable. By increasing W_T and decreasing W_S by a factor of $k > 2$ one gets a stable central controller for $\gamma = 2$, as illustrated by

Figure 13. However, this decreases overall performance, and increases response to noise.

4 Existence of a stable suboptimal controller

The previous section was devoted to studies on the stability of the central controller for a given performance level γ. Sometimes the central controller may be unstable, but there may exist another controller which is stable, and achieves the same performance level γ. In Sections 4.1 and 4.2 this situation will be illustrated with the gust alleviation and tracking examples. But first some possible ways to find such a controller are described.

Recall that, for a given $\gamma > \gamma_{opt}$, all suboptimal controllers are parametrized by (6–8) where $q = Q(s)r$, with $Q \in \mathcal{H}^\infty$ and $\|Q\|_\infty < \gamma$. That is Q is the free parameter, but it has to be stable, and its norm has to be bounded by γ. Suppose that Q has a state space realization

$$\begin{aligned} \dot{x}_q(t) &= A_q x_q(t) + B_q r(t) \\ q(t) &= C_q x_q(t) + D_q r(t) \,. \end{aligned}$$

Then, inserting $q(t)$ into (6) and (7), and using $r(t)$ given by (8), one gets the following realization for the controller (which is a transfer function from $\hat{y}$ to $\hat{u}$)

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_q(t) \end{bmatrix} = \begin{bmatrix} A_c + B_{2c} D_q C_{2c} & B_{2c} C_q \\ B_q C_{2c} & A_q \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_q(t) \end{bmatrix} + \begin{bmatrix} B_{1c} + B_{2c} D_q \\ B_q \end{bmatrix} \hat{y}(t) \tag{19}$$

$$\hat{u}(t) = \begin{bmatrix} C_{1c} + D_q C_{2c} & C_q \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_q(t) \end{bmatrix} + D_q \hat{y}(t) \,. \tag{20}$$

Note that the A-matrix of this controller is

$$A_K := \begin{bmatrix} A_c + B_{2c}D_qC_{2c} & B_{2c}C_q \\ B_qC_{2c} & A_q \end{bmatrix} . \tag{21}$$

Hence, there exists a stable suboptimal $\mathcal{H}^\infty$ controller for a given performance level γ if and only if there exists a "subcontroller" $Q(s)$, stabilizing the "subplant" $G_c(s) := B_{2c}(sI - A_c)^{-1}C_{2c}$, such that $Q \in \mathcal{H}^\infty$ and $\|Q\|_\infty < \gamma$. Thus, testing whether there exists a stable suboptimal $\mathcal{H}^\infty$ controller amounts to checking if G_c is *strongly stabilizable by a controller whose norm is bounded by* γ. In other words, it is necessary (but not sufficient) that G_c satisfy the parity interlacing property. If G_c satisfies p.i.p., then the next question is: can we find a norm bounded stabilizing controller? To the authors' knowledge there have not been any published results on this problem.

One possible way to approach this problem is to apply any stable controller design procedure (see e.g. [1, 5, 4]) to $G_c(s)$, and check whether this controller's ∞-norm is bounded by γ. Obviously, there is no guarantee that this method will work. A rigorous approach would be to study different stable controller design algorithms, and try to determine if it is possible to modify them so that the ∞-norm of the resulting controller is bounded by γ.

Another approach, which is more direct, would be to study the structure of all controllers stabilizing G_c, and determine ways to choose the free parameter so that the controller is stable and norm bounded. To illustrate this method assume that G_c is SISO, then all stabilizing controllers are in the form [1]

$$Q = \frac{X_G + D_G Q_Q}{Y_G - N_G Q_Q} = \frac{D_G(X_G + D_G Q_Q)}{1 - N_G(X_G + D_G Q_Q)}$$

where $G_c = N_G/D_G$ and $N_G, D_G, X_G, Y_G \in \mathcal{H}^\infty$ satisfy the Bezout equation

$$N_G X_G + Y_G D_G = 1 \ ,$$

and $Q_Q \in \mathcal{H}^\infty$ is the free parameter. So, in order to have $Q \in \mathcal{H}^\infty$ and $\|Q\|_\infty < \gamma$, one should try to make $\|X_G + D_G Q_Q\|_\infty$ as small as possible. However, by the Bezout equation X_G is equal to $1/N_G$, at the zeros of D_G in the right half plane. So there is a limit how small this expression can be made. Because of time limitations the authors did not further investigate this method of trying to select an appropriate $Q_Q \in \mathcal{H}^\infty$. It may or may not give satisfactory results.

Note that, when the realization (19-20) is minimal, a dynamic $Q(s)$ (i.e. $Q(s)$ is non-constant) leads to a controller whose order is larger than the order of the generalized plant. This may be an undesirable situation. Therefore, one may want to restrict the search to non-dynamic Q's, in other words $Q(s) = D_q$ which is a constant. In this case, Q is automatically in $\mathcal{H}^\infty$, so the only restriction is on the norm, that is $\sigma_{max}(D_q) < \gamma$. Moreover, if the number of inputs $\hat{y}$ and outputs $\hat{u}$, of G_c, is small then a parameter search, in the space of the entries of D_q, can be done to find a feasible D_q. In the next section this approach will be illustrated on gust alleviation and tracking examples defined earlier.

4.1 Example: gust alleviation problem

Consider the $\mathcal{H}^\infty$ control problem defined in Section 2.1.3, with the parameters $R_1 = R_2 = R_3 = K_1 = K_2 = 1$ and $\sigma = 5$. For this problem $18 < \gamma_{opt} < 19$; and for $\gamma = 25$ the central controller is unstable, with a right half plane pole near $+0.3$. Now suppose $\gamma = 25$ fixed. Then, the problem is to determine if there exists $D_q = [D_{q1} \;\; D_{q2}]$ with

$$D_{q1}^2 + D_{q2}^2 < 25^2 \tag{22}$$

such that the eigenvalues of $A_K = A_c + B_{2c} D_q C_{2c}$ are in the open left half plane. The region in the D_q-plane which gives an affirmative answer to this question is the shaded area in Figure 14, which falls in the circle (22). Note that allowable D_q's are in the form

$$D_q = R\,[\cos\theta \quad \sin\theta]$$

where $0 \leq R < 25$ and $-\pi \leq \theta < \pi$. The plot of $\max \mathrm{Re}\lambda(A_K)$, as R varies between 20 and 25, are as shown in Figure 15, for two cases of fixed $\theta = -10°$ and $\theta = -26°$. Also, the plot of $\max \mathrm{Re}\lambda(A_K)$, as θ varies between $-40°$ and $-28°$, are given in Figure 16, for fixed R at 23.5 and 24.9.

Figures 15 and 16 show that θ should be small and R should be large in order to make A_K stable. Besides the stability of A_K, it is necessary to check the overall system performance in the frequency domain, as well as in the time domain, in order to determine the optimal value of D_q. To illustrate the overall system performance allow to be set equal to $D_q = [24.75 \;\; 0.00]$. As expected the closed loop performance is within specified level γ, i.e. $\sigma_{max}(T_{zw}(j\omega))$ is below

$\gamma = 25$, see Figure 17. For this system, the effects of a pulse gust (i.e. $w_1(t) = w_2(t)$ are equal to a unit pulse of duration 0.25sec) on $z_1, \ldots, z_4$ are shown in Figure 18.

4.2 Example: tracking problem

For the tracking problem defined in Section 2.1.2, recall that $1.5 < \gamma_{opt} < 2$, and for $\gamma = 2$ the central controller generated by the algorithm given in [9] is unstable. Now consider the problem of determining if there exists $D_q = [D_{q1} \;\; D_{q2}]$ such that $A_K = A_c + B_{2c} D_q C_{2c}$ has all its eigenvalues in the open left half plane, and

$$D_{q1}^2 + D_{q2}^2 < 2^2. \tag{23}$$

The shaded region in Figure 19 which falls in the circle (23) indicates possible values of allowable D_q which make A_K stable. Note that compared to previous example, we don't have much freedom in choosing D_q, it has to be chosen close to $D_q = [-1.99 \;\; 0.00]$ in order to get a stable A_K.

In the remainder of this section the performances of the suboptimal $\mathcal{H}^\infty$ controller obatined by choosing $Q(s) = D_q = [-1.99 \;\; 0.00]$, and the unstable central controller, i.e. the one with $D_q = [0 \;\; 0]$, will be compared. Frequency domain closed loop performances are compared in Figure 20. As expected both controllers yield $\|T_{zw}\|_\infty < 2$.

It should also be mentioned that the observation noise $K_2 w_2$ is added to our original $\mathcal{H}^\infty$ control problem definition in order to satisfy rank conditions required in the solution procedure given in [9]. Also, the weights are added to the standard $\mathcal{H}^\infty$ control problem in order to obtain a controller, which gives desired loop shapes; i.e. the weights are not physically present in the system. So, once a controller is obtained it can be implemented on the physical plant as shown in Figure 21. In this figure $\alpha_c(t)$ represents the reference (command) input to the controller, which generates u. The signal u is an input to the actuator, whose output u_p is the plant input. As before, plant outputs are α and q_p.

Time domain performances of the controllers obtained for $\gamma = 2$, with $D_q = [-1.99 \;\; 0.00]$ and with $D_q = [0 \;\; 0]$, are shown in Figure 22. It is seen that the response $\alpha(t)$ is better for $D_q = [0 \;\; 0]$ (i.e. better tracking of a negative unit step $\alpha_c(t)$), and the command signals $u_p(t)$ are similar for both cases. Frequency domain performance plots are shown in Figures 23 and 24, where individual Bode plots (from α_c to $u_p(t)$ and $\alpha(t)$ respectively) are shown. As expected the main difference in the frequency domain characteristics are in the high frequency behavior. Note that when $D_q \neq 0$ the controller is not strictly proper, so compared to $D_q = 0$ there is less high frequency roll-off.

5 Conclusions

In this research possible ways of obtaining stable controllers, achieving a certain pre-specified $\mathcal{H}^\infty$ performance level γ, are identified. First the central controller A_c is made stable by scaling the weights and/or modifying the measured outputs. Secondly, existence of a stable suboptimal controller is investigated by adjusting the free parameter $Q(s)$ which appears in the characterization of all suboptimal controllers. In the first method, since the structure of the $\mathcal{H}^\infty$ problem is modified, the performance of the stable controller may be different than the original unstable controller. In the second method, it is expected that if γ is "close" to its optimal value, then *all* suboptimal controllers are expected to be "close" to the central controller, see e.g. [15]. Therefore, if the central controller is unstable and if γ is close to γ_{opt}, there may not exist a stable suboptimal controller. So, for the second approach to work γ has to be "sufficiently larger" than γ_{opt}.

The second approach leads to an interesting problem: given an unstable plant $G_c(s)$, find a strongly stabilizing controller $Q(s)$ such that $\|Q\|_\infty < \gamma$. This problem is difficult to solve in complete generality. Therefore the authors have concentrated their efforts in demonstarting the existence of a *constant* norm bounded $Q(s) = D_q$ which stabilize $G_c(s)$, for both gust alleviation and tracking problems. The approach here was simple brute-force. Since in this case the number of inputs and outputs, for the controller, was 2 and 1 respectively, the parameter space was two dimensional; and it was easy to check sufficiently large number of points to get an idea about the region in the D_q-space that gives a solution.

Recall that the controller A-matrix, when D_q is used, is given by $A_K = A_c + B_{2c}D_qC_{2c}$, and from (9–13) $A_c = A_X - ZYC_2^TC_2$, $B_{2c} = ZB_2$, and $C_{2c} = -C_2$. Therefore,

$$A_K = A_X - Z(YC_2^T + B_2D_q)C_2, \tag{24}$$

where A_X is stable. This structure might give an idea about how D_q should be chosen for more complicated input/output definitions. For example, one may try to minimize the norm of $Z(YC_2^T + B_2D_q)C_2$. This is a relatively easy optimization problem, which can be solved using existing commercially available software. However, there is no guarantee that after minimization all the eigenvalues of A_K will be in the open left half plane.

The effects of weight scalings and modifications in the measured outputs, on stability of the central controller, are studied in the first part of this research. Recall that A_c is in the form (9) or (14). More specifically $A_c = A_X - ZYC_2^TC_2$ or $A_c = A_Y - B_2B_2^TXZ$ where A_Y and A_X are stable matrices. If the measured outputs are modified, then C_2 changes, and without changing A_X it changes the term $ZYC_2^TC_2$. Therefore, one might be able to design C_2 so that $ZYC_2^TC_2$ is "small" compared to the eigenvalues of A_X, and by using Gershgorin's theorem [16] it may be possible to guarantee a stable A_c. Similarly, if control inputs can be redefined, i.e. B_2 can be modified, then one can try to make $B_2B_2^TXZ$ "small" without changing A_Y, and this may lead to a stable controller. It is obvious from (24) that simultaneously modifying C_2 and selecting D_q which minimizes $Z(YC_2^T + B_2D_q)C_2$ increases chances of getting a stable controller whose poles are close to the eigenvalues of A_X.

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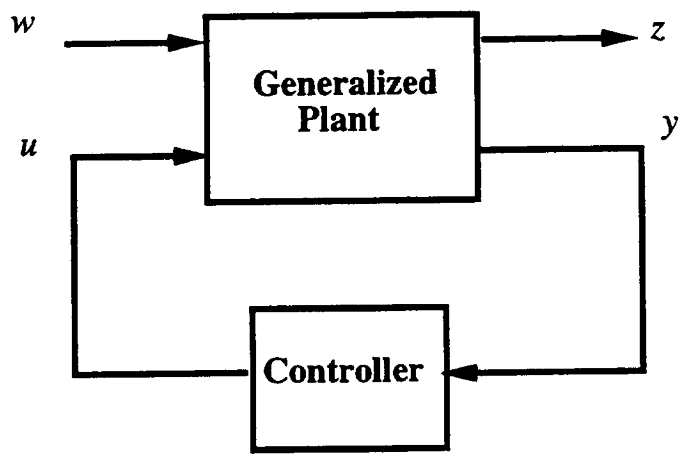

Figure 1: Standard Feedback System

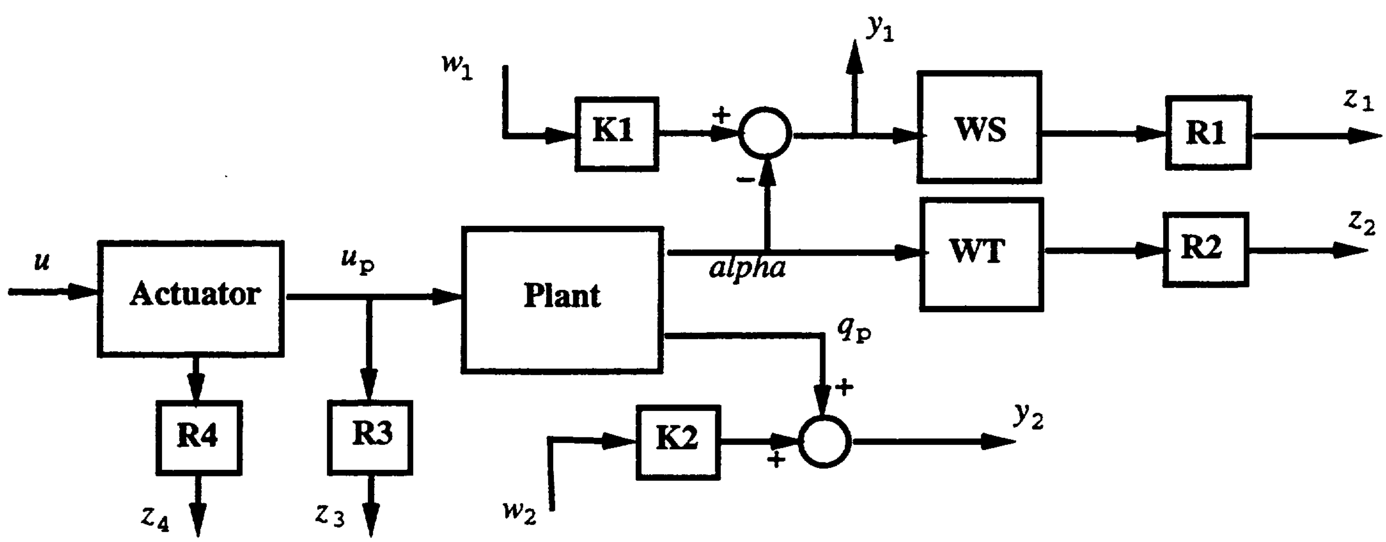

Figure 2: Tracking Problem

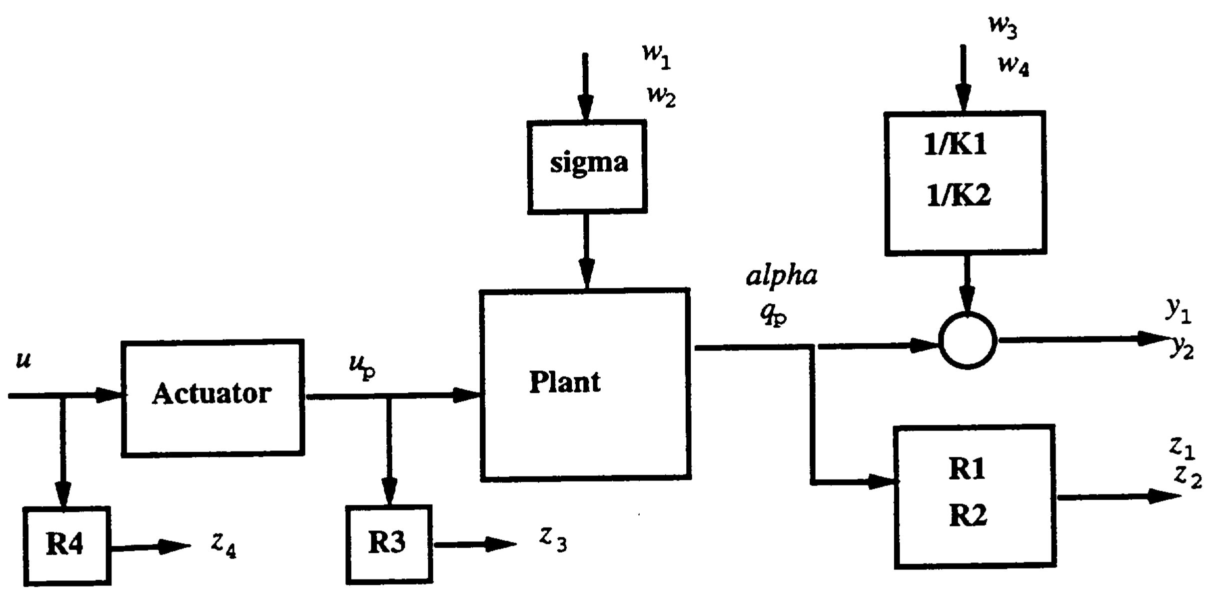

Figure 3: Gust alleviation problem

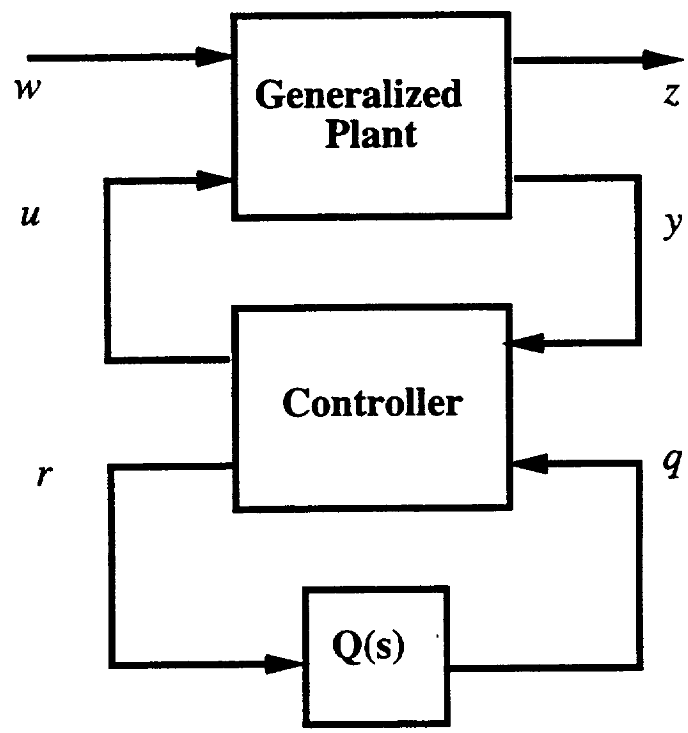

Figure 4: Suboptimal $\mathcal{H}^\infty$ controllers

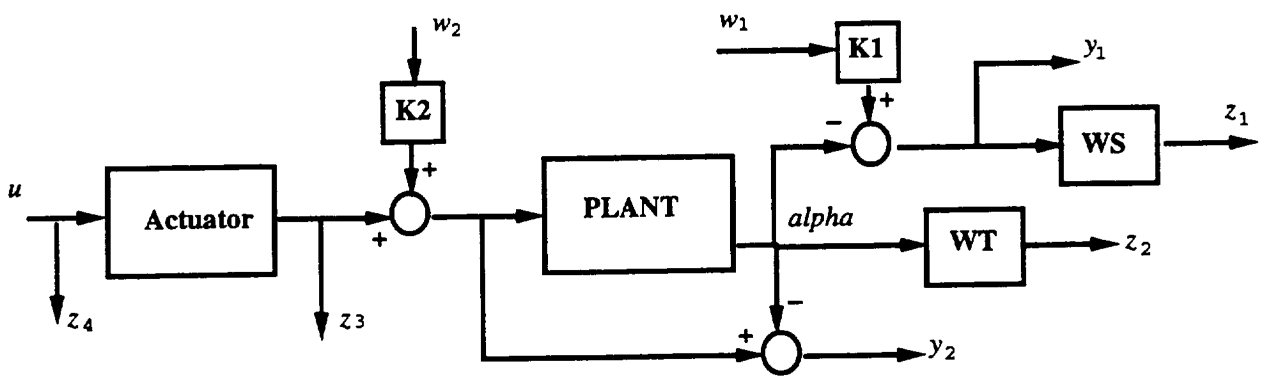

Figure 5: Alternative tracking problem definition

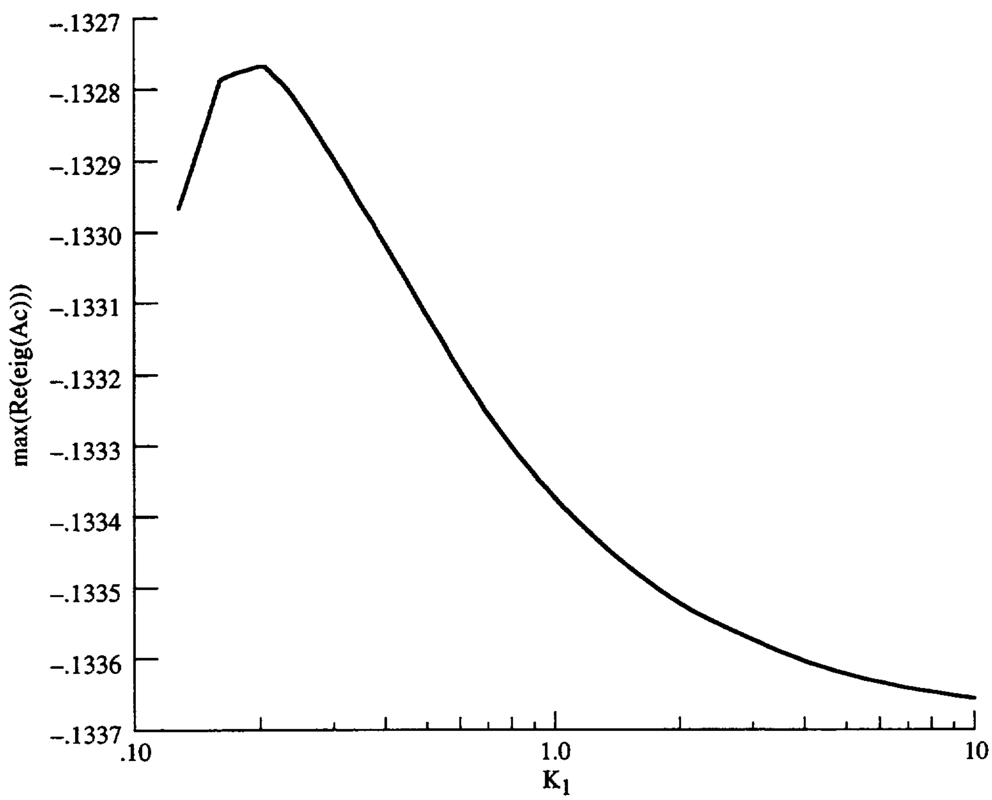

Figure 6: The effect of K_1 on the most unstable eigenvalue of A_c; for $\gamma = 30$

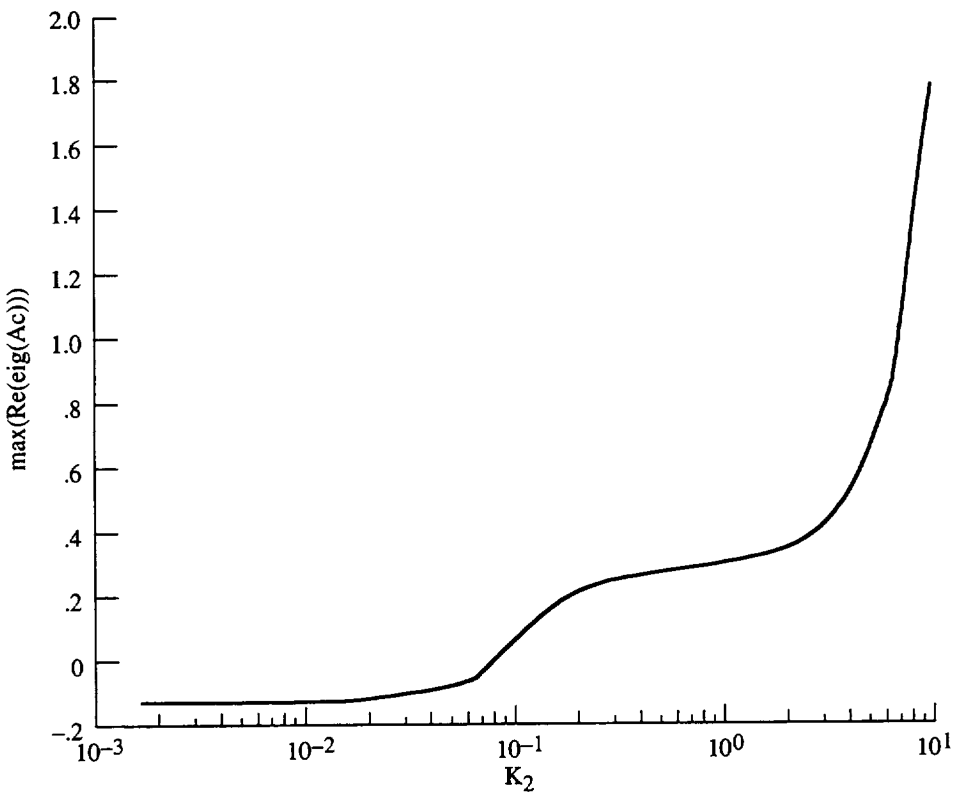

Figure 7: The effect of K_2 on the most unstable eigenvalue of A_c; for $\gamma = 25$

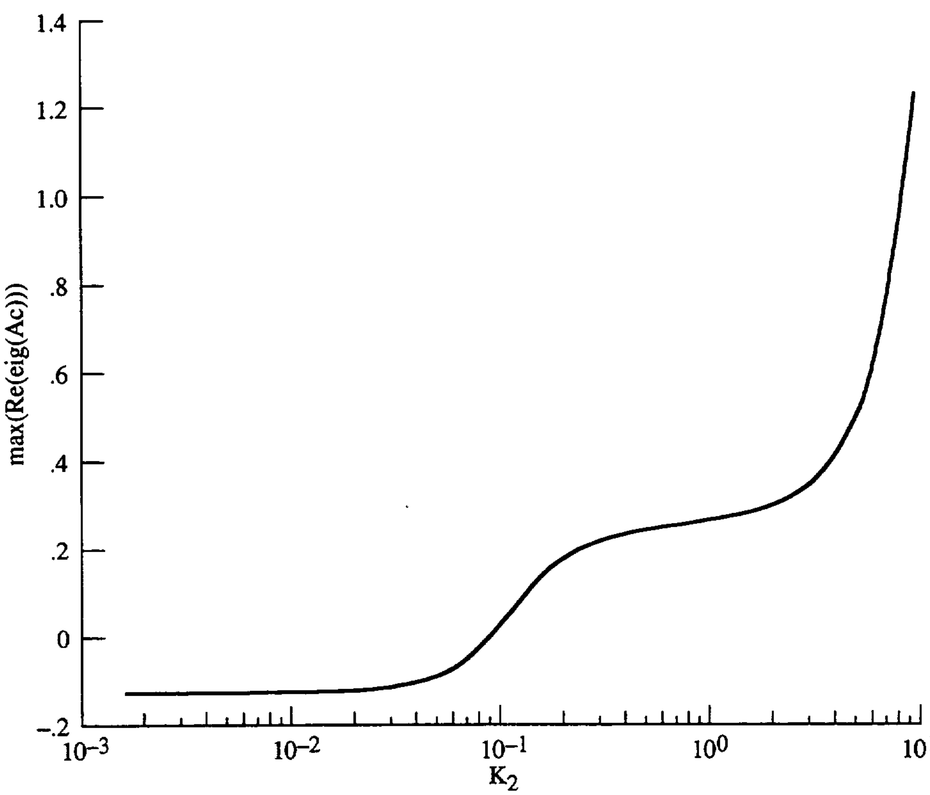

Figure 8: The effect of K_2 on the most unstable eigenvalue of A_c; for $\gamma = 30$

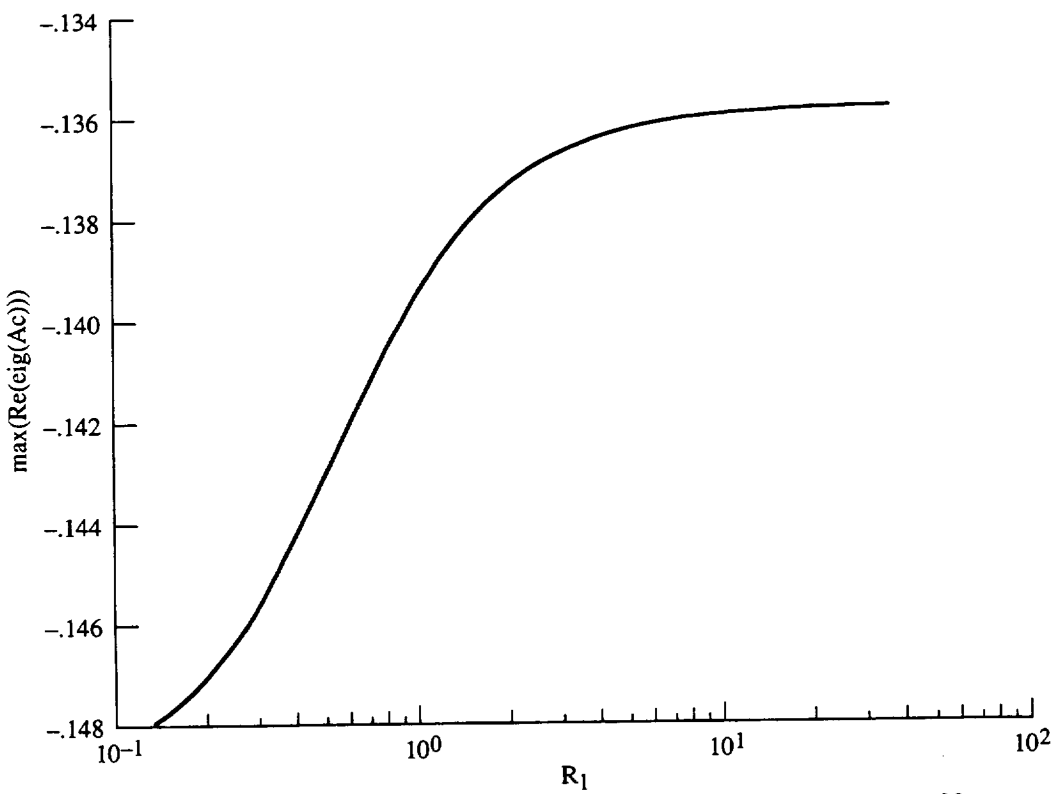

Figure 9: The effect of R_1 on the most unstable eigenvalue of A_c; for $\gamma = 80$

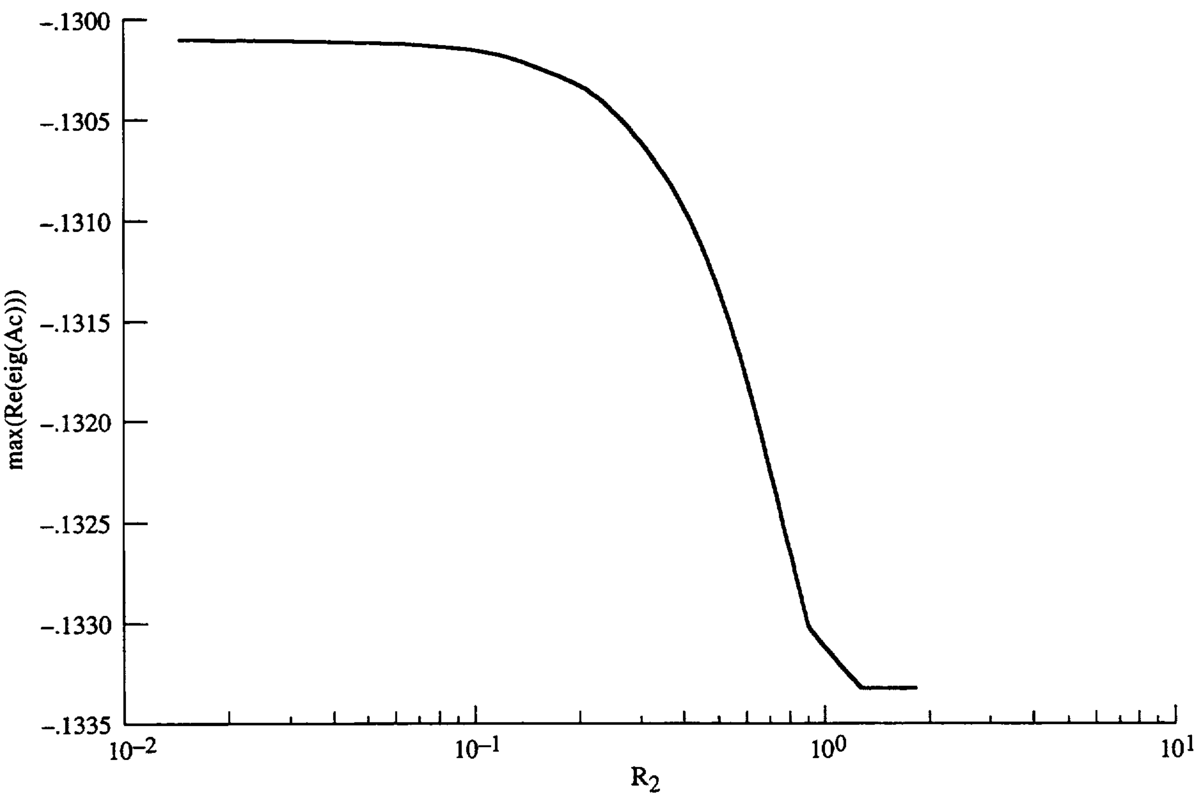

Figure 10: The effect of R_2 on the most unstable eigenvalue of A_c; for $\gamma = 30$

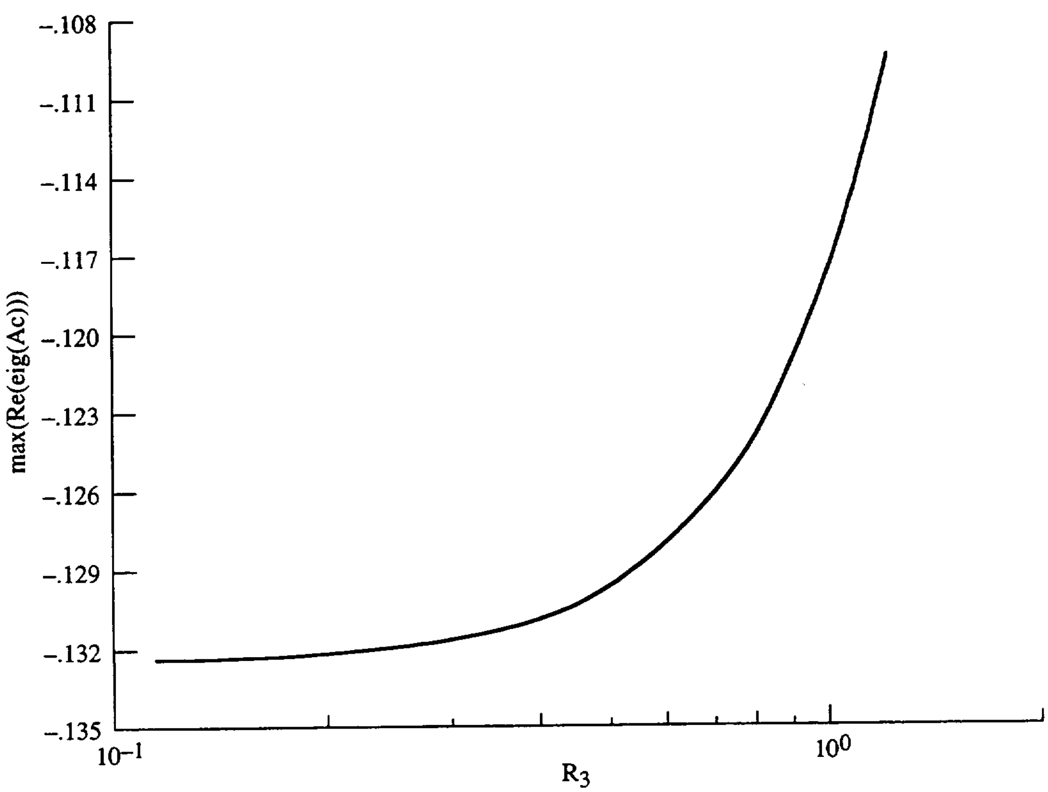

Figure 11: The effect of R_3 on the most unstable eigenvalue of A_c; for $\gamma = 25$

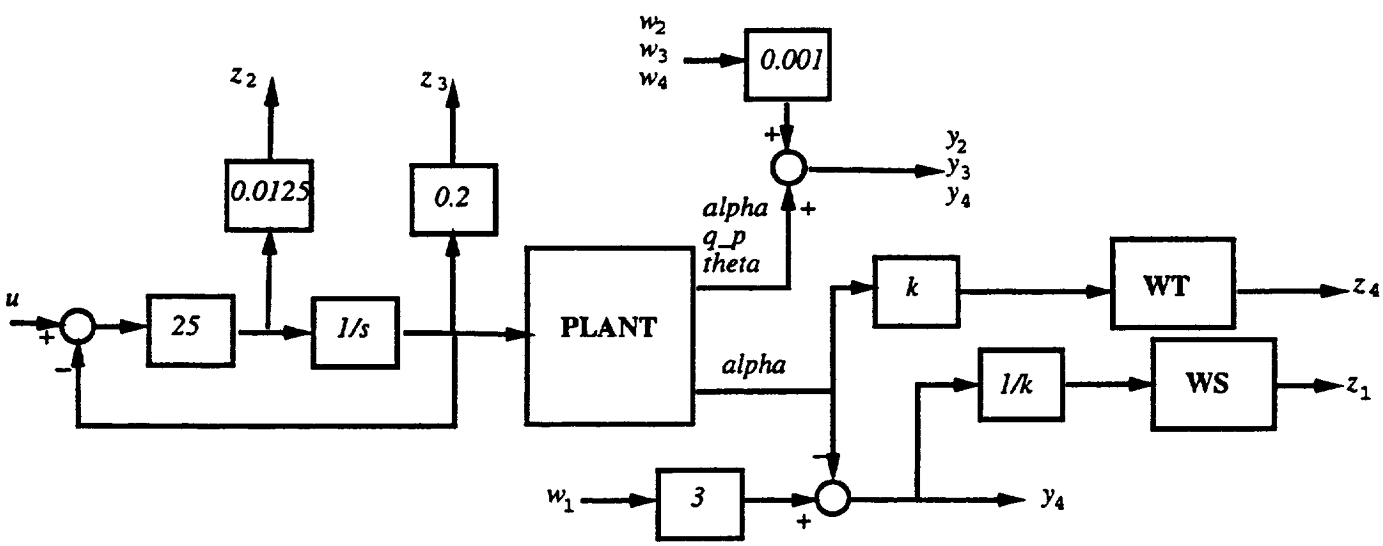

Figure 12: Tracking problem with more outputs, and scaled W_S, W_T

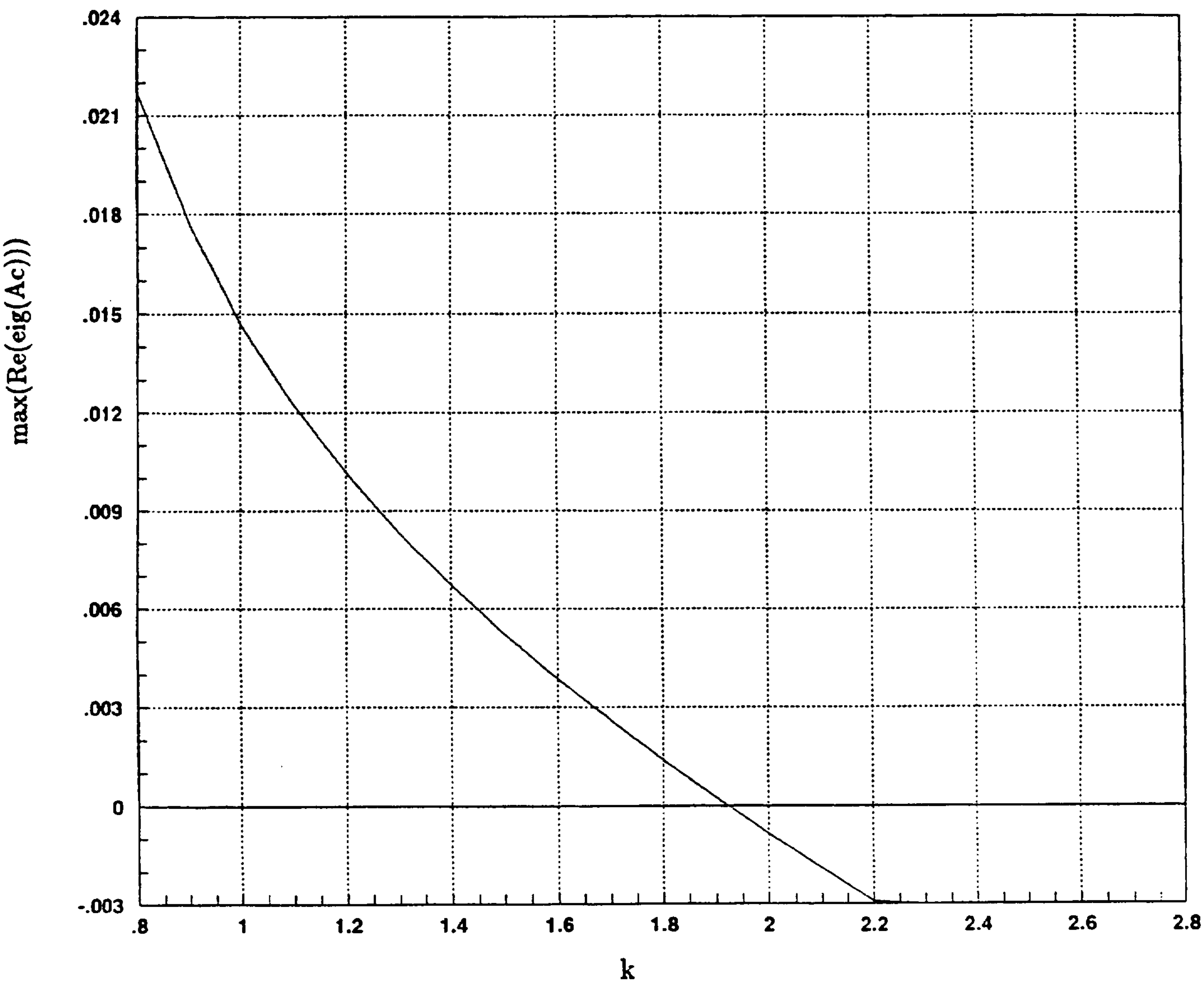

Figure 13: max $\mathrm{Re}\lambda(Ac)$ versus k, for $\gamma = 2$

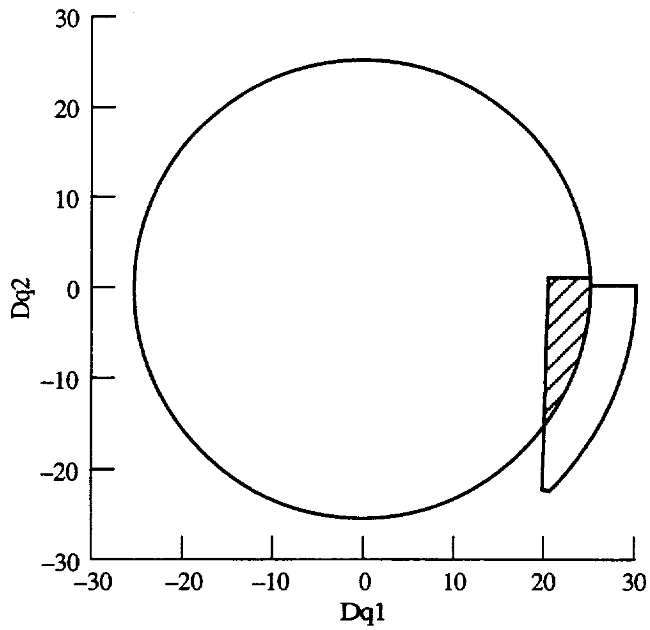

Figure 14: Values of D_q which make A_K stable

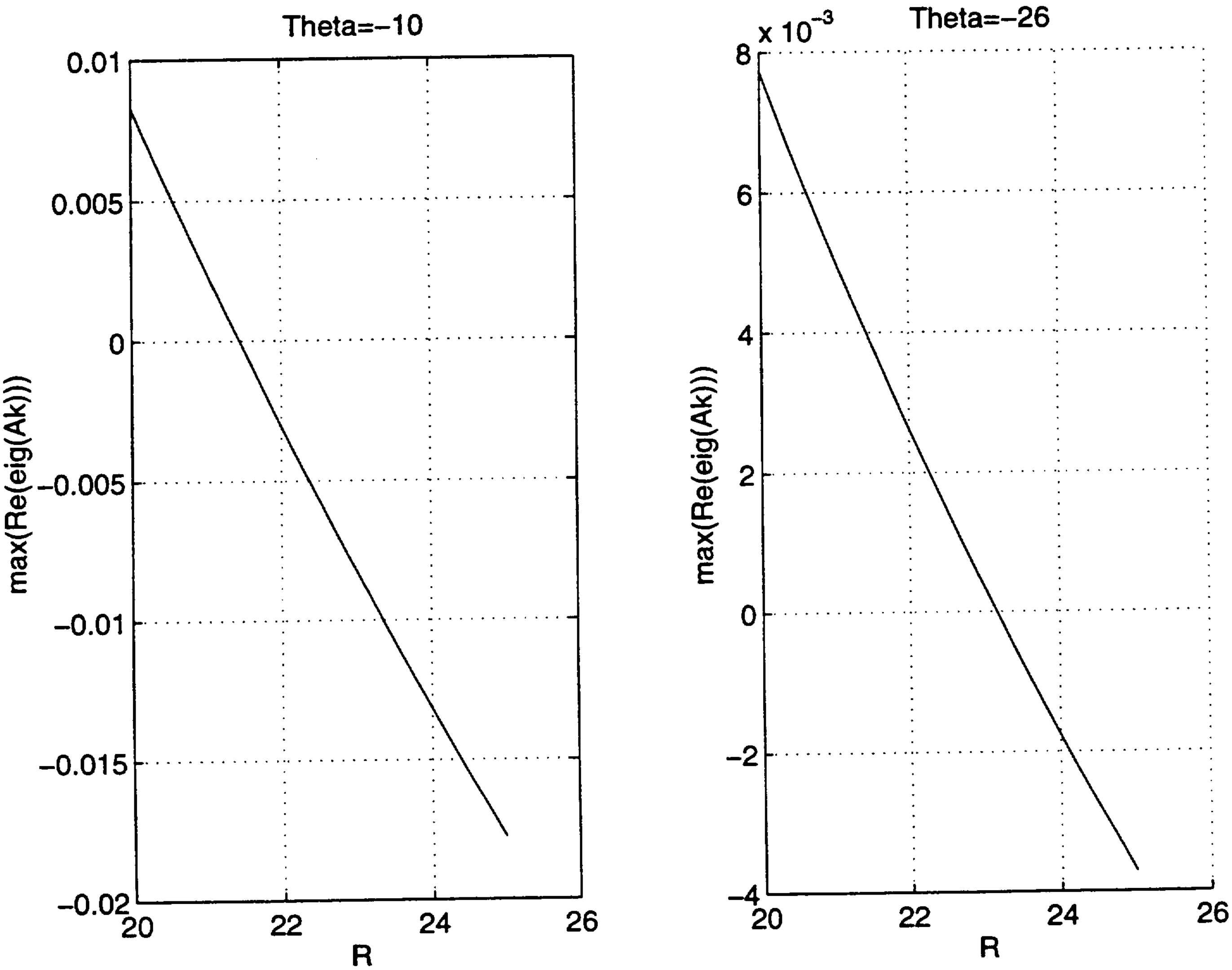

Figure 15: Effect of R on the most unstable eigenvalue of A_K, for $\theta = -10°$ and $-26°$

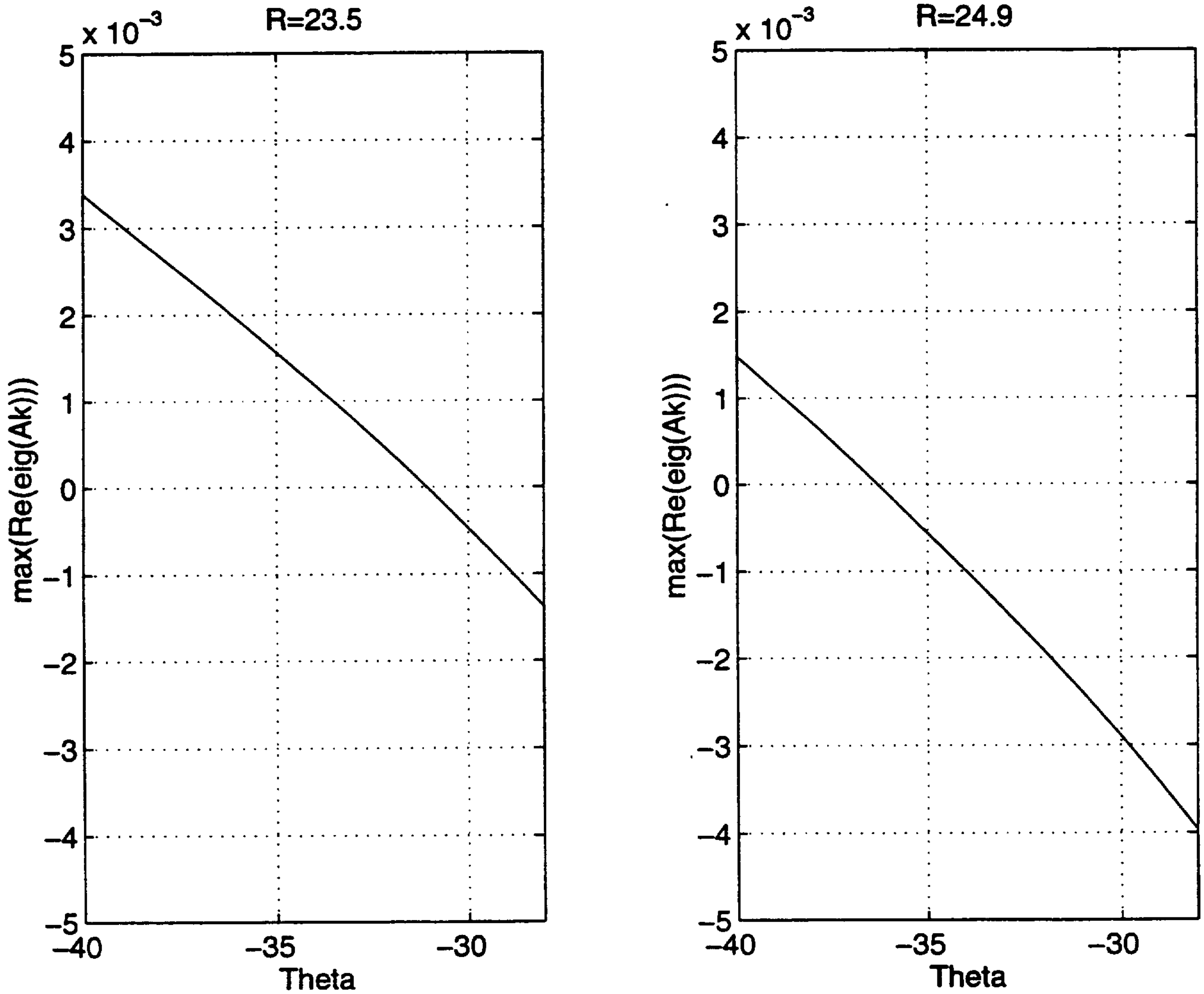

Figure 16: Effect of θ on the most unstable eigenvalue of A_K, for $R = 23.5$ and $R = 24.9$

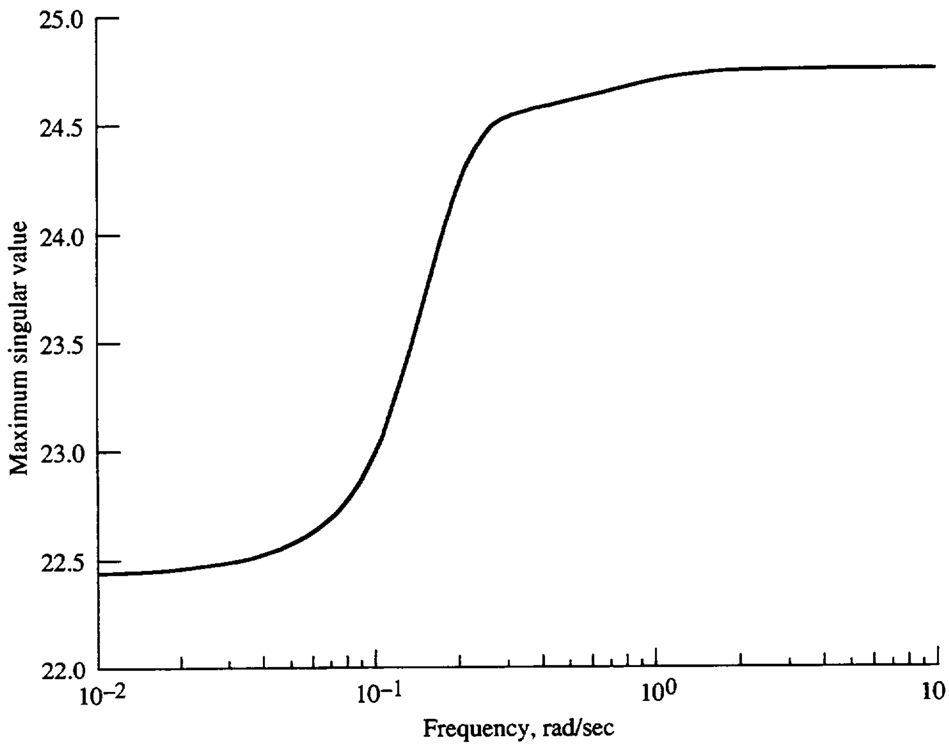

Figure 17: Closed loop performance for the gust alleviation problem; $\alpha_{max}(T_{zw}(j\omega))$ versus ω, for $D_q = [24.75 \;\; 0.00]$

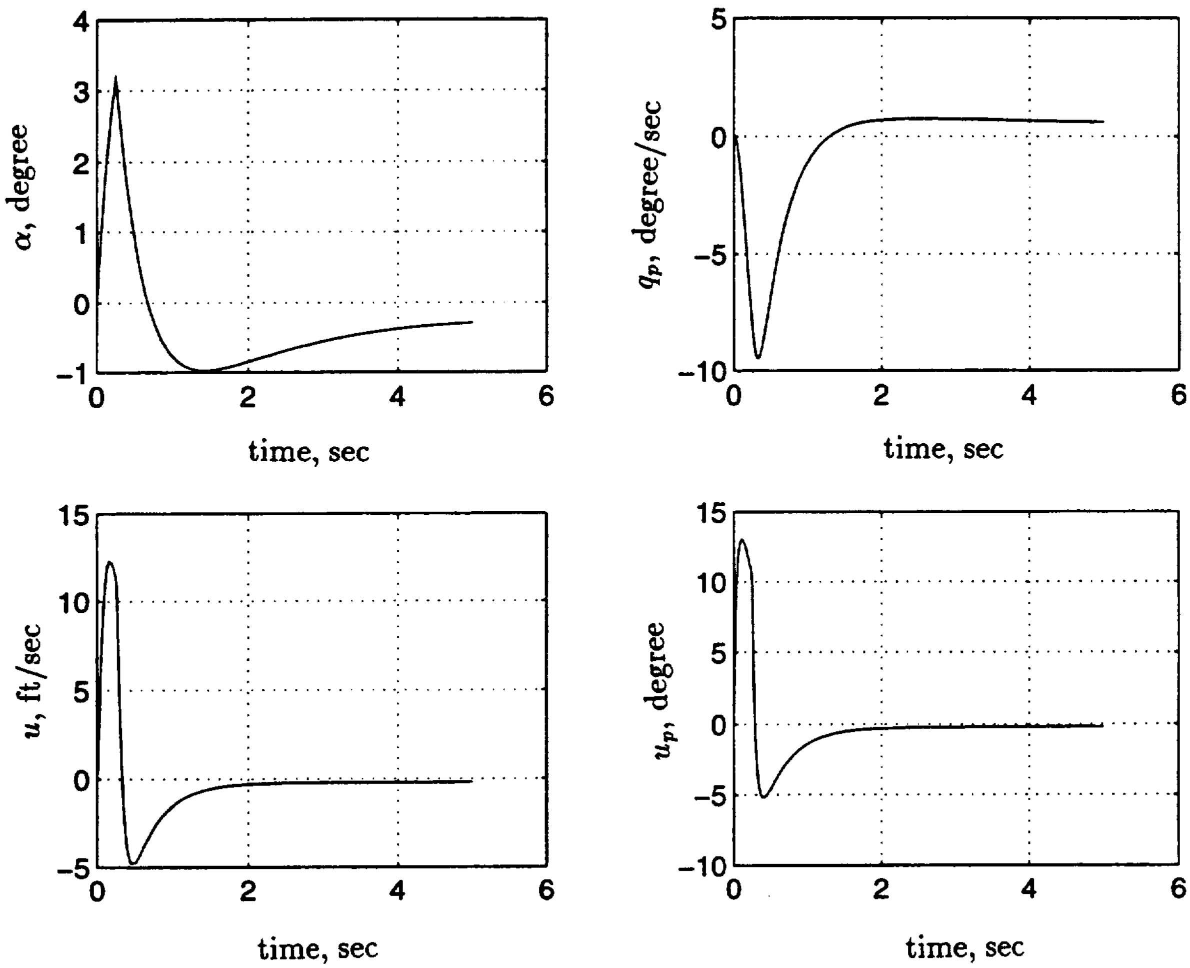

Figure 18: Time resposes to a unit pulse gust, for $D_q = [24.75 \;\; 0.00]$

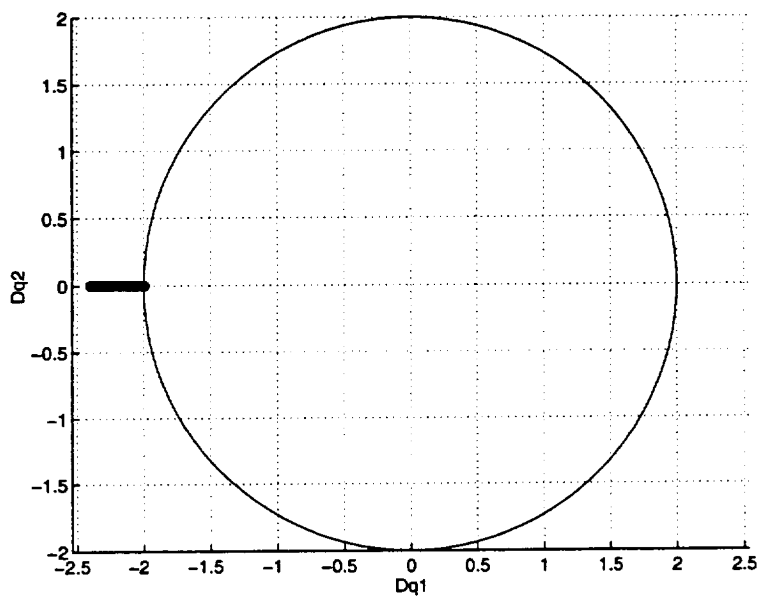

Figure 19: Values of D_q which make A_K stable

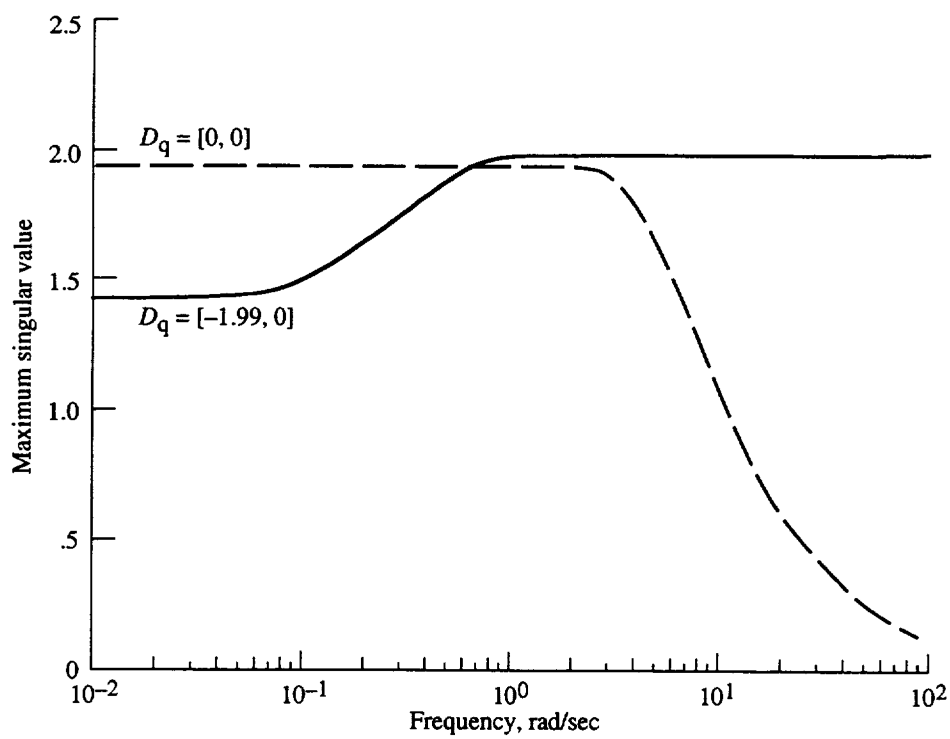

Figure 20: Closed loop performance for tracking problem; $\alpha_{max}(T_{zw}(j\omega))$ versus ω, for $D_q = [-1.99\ \ 0.00]$, and for $D_q = [0\ \ 0]$

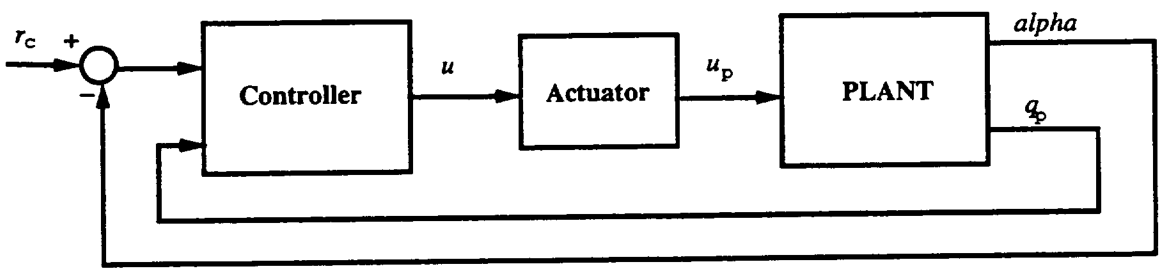

Figure 21: Controller implementation on the physical plant

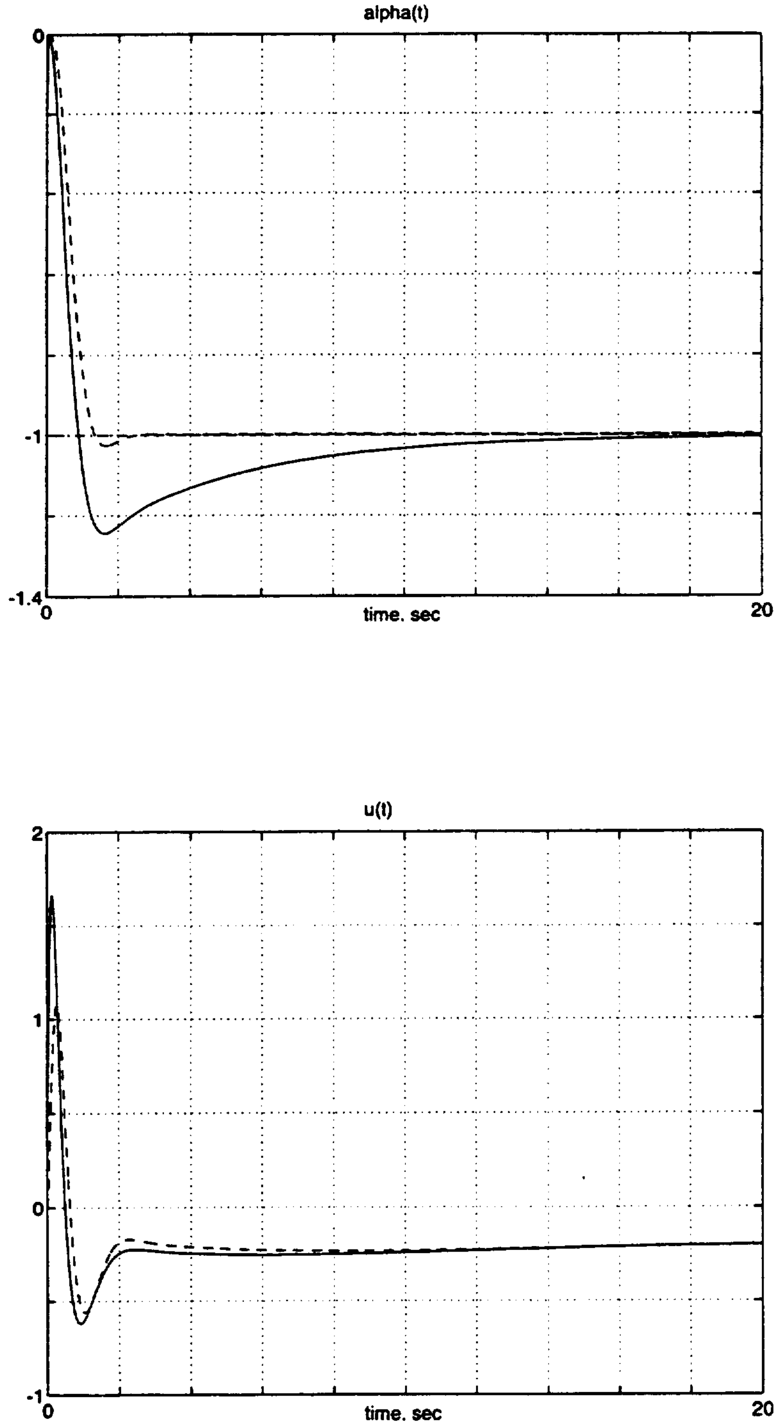

Figure 22: Time responses to a negative unit step $r_c(t)$;

$Dq = [-1.99 \;,\; 0]$ (solid line), $Dq = [0 \;,\; 0]$ (dashed line)

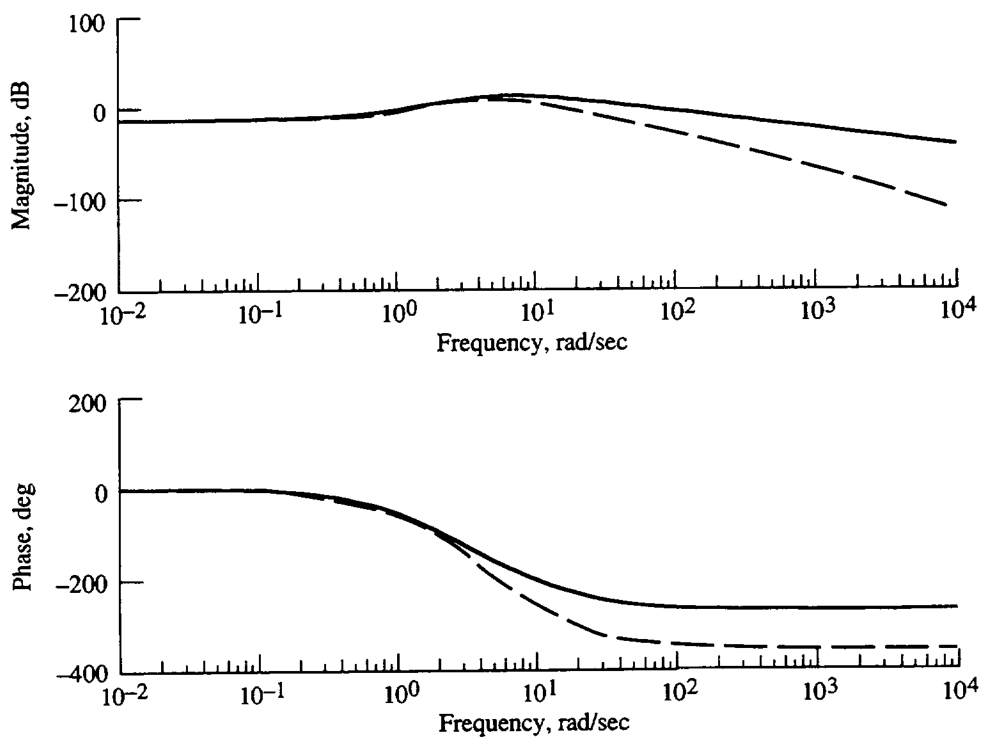

Figure 23: Bode plot of the closed loop transfer function from α_c to u_p;
$D_q = [-1.99, 0]$ (solid line), $D_q = [0, 0]$ (dashed line)

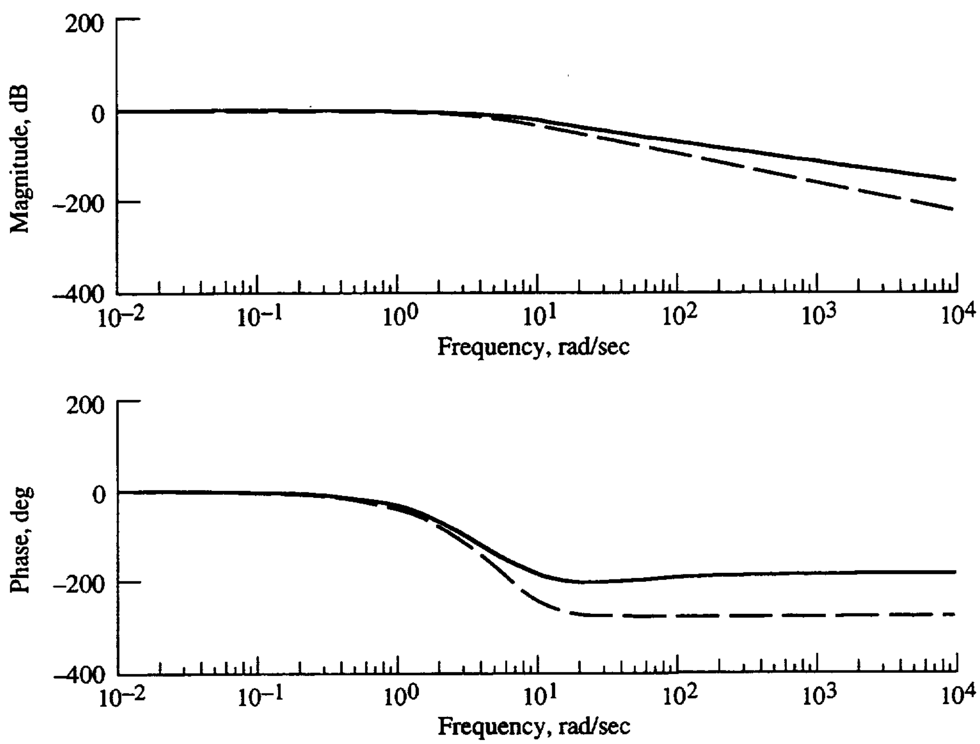

Figure 24: Bode plot of the closed loop transfer function from α_c to α; D_q = [−1.99, 0] (solid line), D_q = [0, 0] (dashed line)

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13. ABSTRACT ***(Maximum 200 words)***
This report discusses different approaches to stable H^∞ controller design applied to the problem of augmenting the longitudinal dynamics of an aircraft. Stability of the H^∞ controller is investigated by analyzing the effects of changes in the performance index weights, and modifications in the measured outputs. The existence of a stable suboptimal controller is also investigated. It is shown that this is equivalent to finding a stable controller, whose infinity norm is less than a specified bound, for an unstable plant which is determined from parametrization of all H^∞ controllers. Examples are given for a gust alleviation and a command tracking problem.

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